Energy spectral density

Definition of inner/dot product:

$$ <x,v> = \sum_{k = -\infty}^{\infty} x[k]v[k] $$

We can then calculate the dot product of a signal and a time-shifted signal. Let's call that dot product, which is a function of n, $p[n]$.

$$ p[n] = \sum_{k = -\infty}^{\infty} x[k]v[k-n] $$

This formula can be rewritten as a convolution by defining a new function $\overleftarrow{v}[n] \equiv v[-n]$. This is the time-reversed version of the $v[n]$.

$$ p[n] = \sum_{k = -\infty}^{\infty} x[k]\overleftarrow{v}[n-k] = (x \ast \overleftarrow{v})[n] $$

The convolution in the time domain is equivalent to multiplication in the frequency domain.

$$ P(e^{j\Omega}) = \sum_{k = -\infty}^{\infty} p[k] e^{-j\Omega k} $$ $$ = X(e^{j\Omega}) \sum_{k = -\infty}^{\infty} \overleftarrow{v}[k] e^{-j\Omega k} $$ $$ = X(e^{j\Omega}) \sum_{k = -\infty}^{\infty} v[-k] e^{-j\Omega k} $$ $$ = X(e^{j\Omega}) V(e^{-j\Omega}) $$

$$ p[0] = \sum_{k = -\infty}^{\infty} x[k]v[k] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\Omega}) V(e^{-j\Omega}) d\Omega$$

In the special case where $v[n] = x[n]$, this dot product $p[n]$ is the deterministic autocorrelation $\bar{R}_{xx}[n]$.

$$ \bar{R}_{xx}[n] = \sum_{k = -\infty}^{\infty} x[k]x[k-n] $$

The Fourier transform of $\bar{R}_{xx}[n]$, a time-domain signal, is the (deterministic) energy spectral density $\bar{S}_{xx}(e^{j\Omega})$

$$ \bar{S}_{xx}(e^{j\Omega}) = |X(e^{j\Omega})|^2 $$

The **energy** of the signal $x[n]$ is the value of the deterministic autocorrelation at $n=0$. This is the result given by Parseval's theorem.

$$ E_x = \bar{R}_{xx}[0] = \sum_{k = -\infty}^{\infty} |x[k]^2| = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{-j\Omega})|^2 d\Omega $$

A signal for which this sum (energy) is finite, or is “square summable,” is denoted as ℓ_2 (ell two).

Consider $y[n]$, which is the result of applying the signal $x[n]$ to some filter with frequency response $H(e^{j\Omega})$. Then,

$$ Y(e^{j\Omega}) = H(e^{j\Omega}) X(e^{j\Omega}) $$

$$ \bar{S}_{xx} = |X(e^{j\Omega})|^2 $$

Cross-spectral density of Y and X:

$$ \bar{S}_{yx} = Y(e^{j\Omega})X(e^{-j\Omega}) $$ $$ = H(e^{j\Omega}) |X(e^{j\Omega})|^2 $$ $$ = H(e^{j\Omega}) \bar{S}_{XX}(e^{j\Omega}) $$

- To calculate the deterministic cross-correlation between the input and output signals, multiply the ESD of the input by the frequency response.

Deterministic cross-correlation of y and x:

$$ \bar{R}_{yx}[n] = \sum_{k = -\infty}^{\infty} y[n]x[k-n] $$

Energy spectral density of $y[n]$:

$$ \bar{S}_{yy}(e^{j\Omega}) = Y(e^{j\Omega})Y(e^{-j\Omega}) = H(e^{j\Omega}) H(e^{-j\Omega}) X(e^{j\Omega}) X(e^{-j\Omega}) = |H(e^{j\Omega})|^2 \bar{S}_{xx}(e^{j\Omega}) $$

- To calculate the energy spectral density of the output signal, multiply the ESD of the input by the magnitude of the frequency response squared.