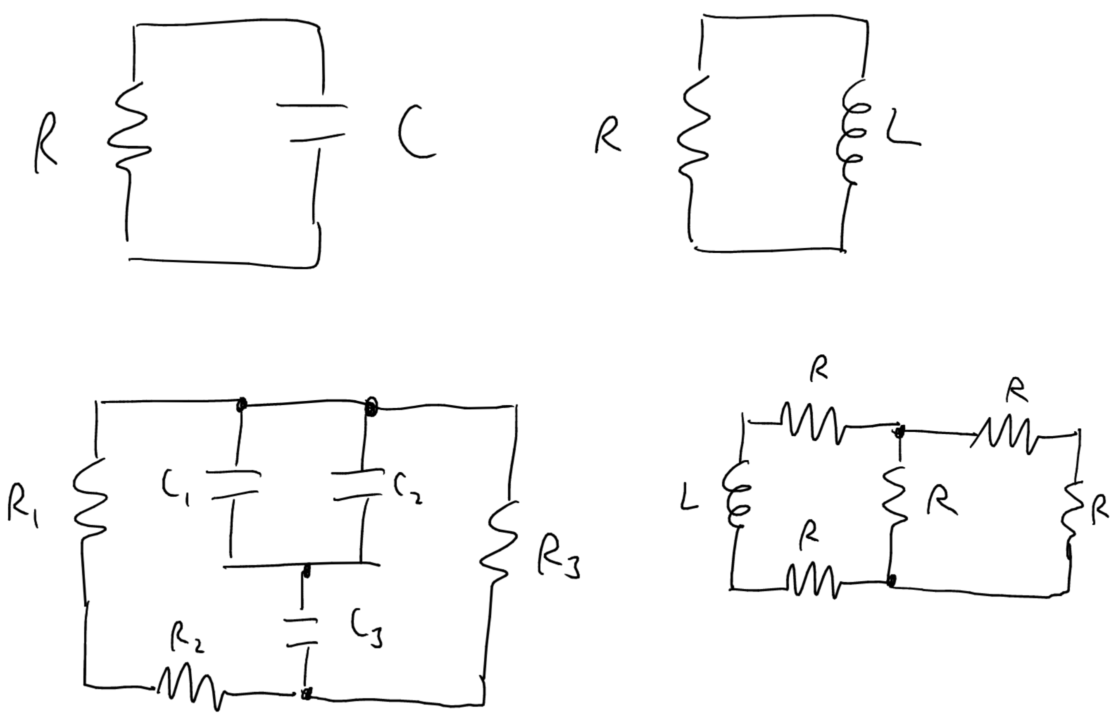
First-order circuits

First-order circuits are circuits that, when all independent sources are turned off, can be simplified to either a circuit with a single resistor and a single capacitor or inductor.

These are all examples of first-order circuits.



Properties of C and L

These properties of capacitors and inductors are useful to know for solving first-order circuits:

Element	**Constitutive relation**	**Initial behavior**	**Final behavior**	**Time constant**
Capacitors	$i_C = C \frac{dv_C}{dt}$	$v_C(0^+) = v_C(0^-)$ - acts like a short **only if** $v_C(0^-) = 0$	$i_C(\infty) = 0$ - acts like an open	$\tau = RC$
Inductors	$v_L = L \frac{di_L}{dt}$	$i_L(0^+) = i_L(0^-)$ - acts like an open **only if** $i_L(0^-) = 0$	$v_L(\infty) = 0$ - acts like a short/wire	$\tau = L / R$

Time response

Many problems involving first-order circuits ask you to find the voltage or current in a first-order circuit as time passes. Usually, the circuit will be in some steady state. Then, something in the circuit will change, and you'll be asked to calculate how some circuit variables evolve over time.

The general procedure for solving these problems involves the following steps:

1. Calculate the initial value of the variable.
2. Calculate the final value of the variable.
3. Find the time constant τ.

In general, I recommend that you **avoid setting up differential equations to solve first-order circuits** unless you are explicitly asked to do so.

The circuit variable will evolve according to the exponential decay function, which looks like this:

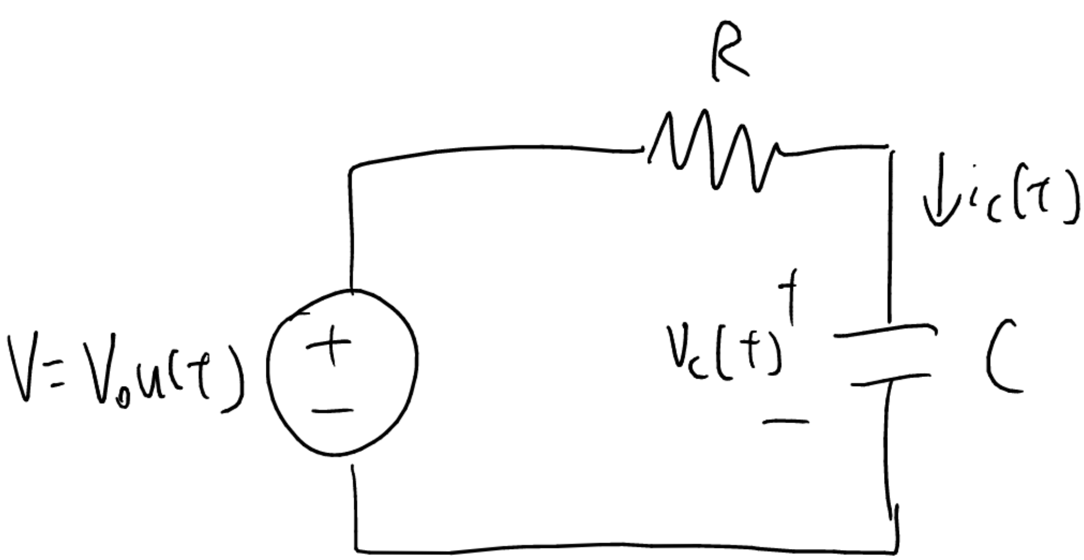
$$ X(t) = X(0^+) e^{-t/\tau} + X(\infty) \left(1 - e^{-t/\tau} \right) $$

Here, $X(t)$ can represent either current $i(t)$ or voltage $v(t)$.

Notice that the term with the “initial value” coefficient ($X(0^+)$) goes from $1 \to 0$ as $t \to \infty$, whereas the term with the “final value” coefficient ($X(\infty)$) goes from $0 \to 1$ as $t \to \infty$.

Let's go through an example involving a step response driving an RC circuit.

Solve for the voltage across and current through the capacitor C.



Initial values

The initial value is the value of the circuit variable right after the change in the circuit. Here, the change in the circuit is in the voltage of the voltage source, which goes from 0 to V_0 at $t = 0$. So the initial current and voltage would be $i_C(0^+)$ and $v_C(0^+)$ respectively. What are $i_C(0^+)$ and $v_C(0^+)$?

Let's start with $v_C(0^+)$. Since the **voltage across a capacitor is continuous,** we know that $v_C(0^+) = v_C(0^-)$. (**For an inductor, the current is continuous**: $i_L(0^+) = i_L(0^-)$.) And at $t = 0^-$, the voltage of the voltage source was 0, so $v_C(0^-) = 0$. Therefore,

$$v_C(0^+) = v_C(0^-) = 0$$

To find $i_C(0^+)$, we can take advantage of the fact that $i_C(t) = i_R(t)$. Due to KVL, The initial values of the circuit variables are the values right after the change occurs.

$$v_R(0^+) = V_0 - v_C(0^+) = V_0$$

And from Ohm's law, we know that:

$$ i_R(0^+) = \frac{V_0}{R} $$

Therefore,

$$ i_C(0^+) = i_R(0^+) = \frac{V_0}{R} $$

Now we have the initial conditions:

$$ v_C(0^+) = 0 $$ $$ i_C(0^+) = \frac{V_0}{R} $$

Final values

The values of the circuit variables when the circuit is in steady state are the final values. To find the steady state of the circuit, let's think back to the constitutive relations of capacitors and inductors:

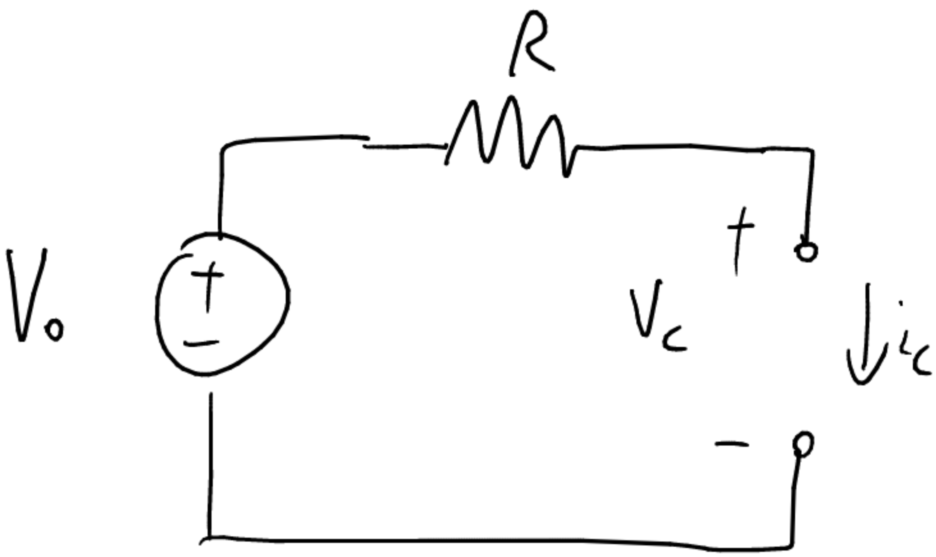
$$ i_C = C \frac{di_C}{dt} $$ $$ v_L = L \frac{dv_L}{dt} $$

When the circuit is in steady state, there is no change in any of its variables. That means all time derivatives of circuit variables are zero. So, in steady state,

$$ i_C(\infty) = C \frac{dv_C}{dt} = 0 $$ $$ v_L(\infty) = L \frac{di_L}{dt} = 0 $$.

In other words, in steady state, **capacitors act like open circuits,** and **inductors act like short circuits.**

Let's go back to our example. To solve for the final values of current and voltage, we just need to replace the capacitor with an open circuit.



Obviously, the current i_C is going to be zero. And because there's no current going through the resistor R, v_C equals V_0.

Time constant

For RC circuits, the time constant is $\tau = RC$, where R is the equivalent resistance of the circuit as seen from the capacitor. Similarly, for RL circuits, the time constant is $\tau = \frac{L}{R}$.

The time constant of this example first-order circuit is $\tau = RC$.

Putting it all together

All we need to do now is plug in the initial value, final value, and time constant into the exponential decay function.

$$ i_C(t) = i_C(0^+) e^{-t/\tau} + i_C(\infty) \left(1 - e^{-t/\tau} \right) = \frac{V_0}{R} e^{-t/(RC)} + (0) \left(1 - e^{-t/(RC)} \right) = \frac{V_0}{R} e^{-t/(RC)}$$ $$ v_C(t) = v_C(0^+) e^{-t/\tau} + v_C(\infty) \left(1 - e^{-t/\tau} \right) = (0) e^{-t/(RC)} + V_0 \left(1 - e^{-t/(RC)} \right) = V_0 \left(1 - e^{-t/(RC)} \right)$$

If we plug in $t=0$ into these functions, we should get the correct initial conditions:

$$ i_C(0) = \frac{V_0}{R} e^{-(0)/(RC)} = \frac{V_0}{R} $$ $$ v_C(0) = V_0 \left(1 - e^{-(0)/(RC)} \right) = V_0 (1 - 1) = 0 $$

OK, initial conditions check out! Now let's try plugging in “$t = \infty$.”Course 18s, please don't get mad at me...

$$ i_C(\infty) = \frac{V_0}{R} e^{-(\infty)/(RC)} = 0 $$ $$ v_C(\infty) = V_0 \left(1 - e^{-(\infty)/(RC)} \right) = V_0 (1 - 0) = V_0 $$

Final conditions check out too.

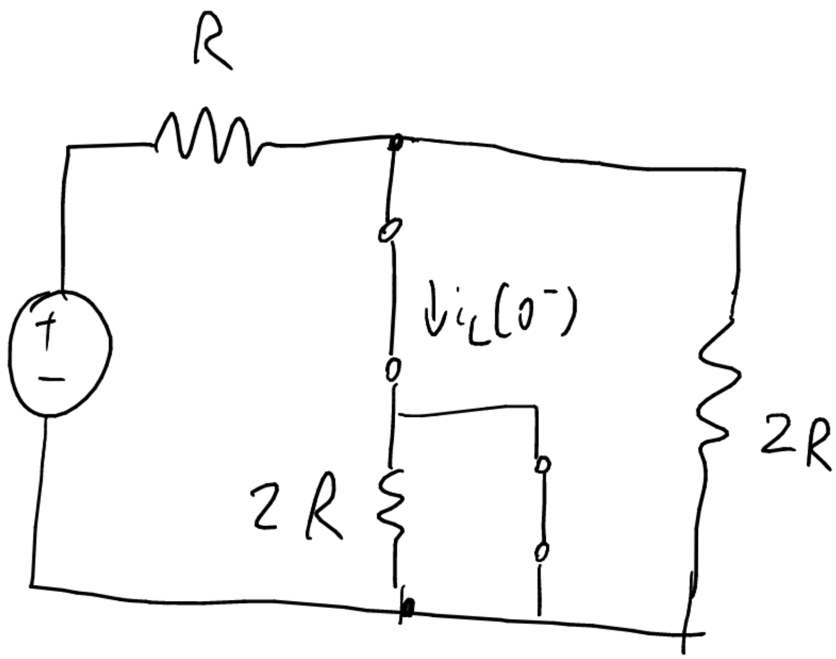
Another example (from Fall 2019 Midterm 2, Problem 3)

That was pretty much the simplest possible example. Let's try a more complicated circuit, this time with an inductor.

Assume the switch has been closed for a long time. At $t=0$, the switch is opened. Find the current through the inductor $i_L(t)$ and voltage across the inductor $v_L(t)$ as functions of time.

Initial values

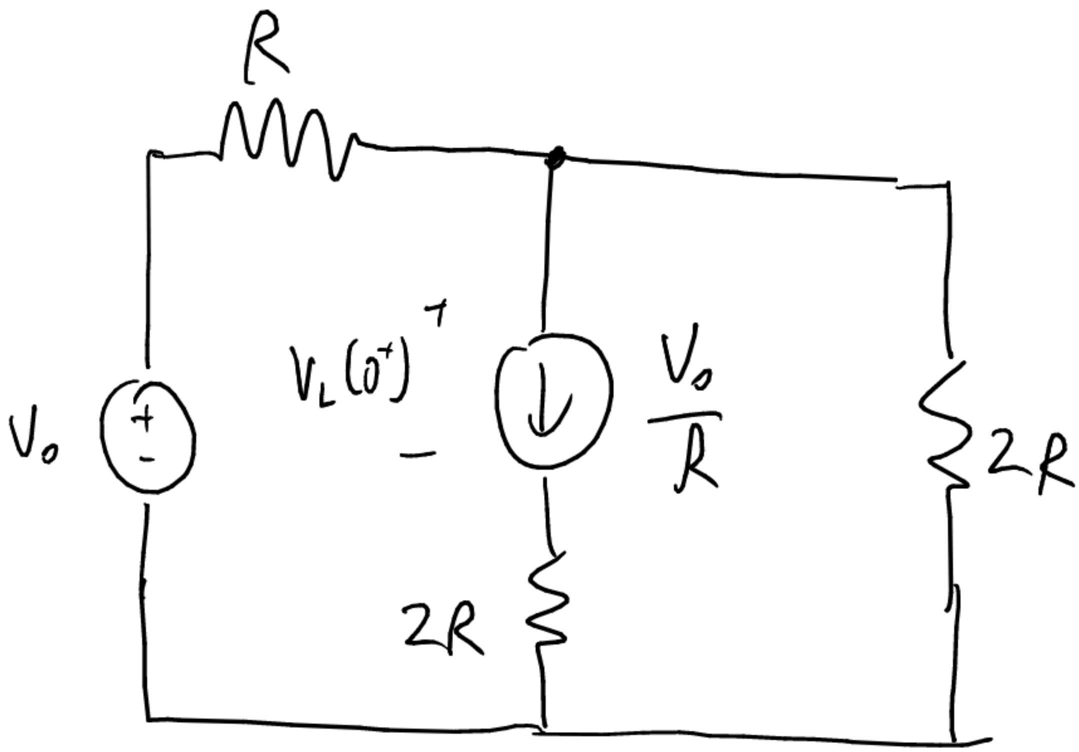
The initial values of the circuit variables are the values right after the change occurs. Let's start with the current through the inductor. Since the **current through an inductor has to be continuous,** $i_L(0^+)$, or the current right after the switch is opened, is equal to $i_L(0^-)$. To find $i_L(0^-)$, we need to solve for this current **before the switch was opened.** We stated that the switch has been closed for a long time, so we can assume that the circuit was in steady state. So, the voltage across the inductor $v_L(0^-)$ was zero, meaning we can replace it with a short circuit.



From this circuit, we learn that:

$$i_L(0^+) = i_L(0^-) = \frac{V_0}{R}$$

For the voltage across the inductor, we have to use the $i_L(0^+)$ that we just solved for. Since $i_L(0^+) = \frac{V_0}{R}$, we can replace the inductor with a current source with $I = \frac{V_0}{R}$ **for the instant $t=0^+$.**



From this circuit, we learn that:

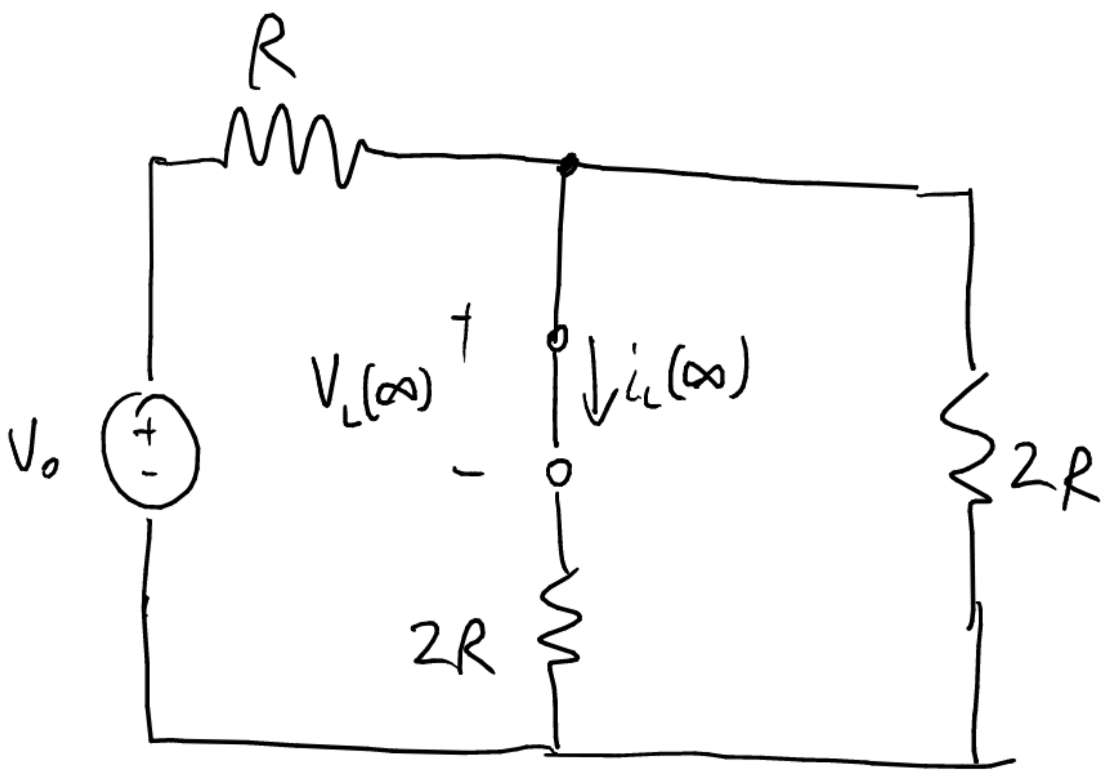
$$ v_L(0^+) = -2V_0 $$

Now we have the initial values:

$$i_L(0^+) = \frac{V_0}{R}$$ $$ v_L(0^+) = -2V_0 $$

Final values

To find the final values, we have to solve for the steady state of this circuit when the switch is open. Just as before, the inductor can be replaced with a wire.



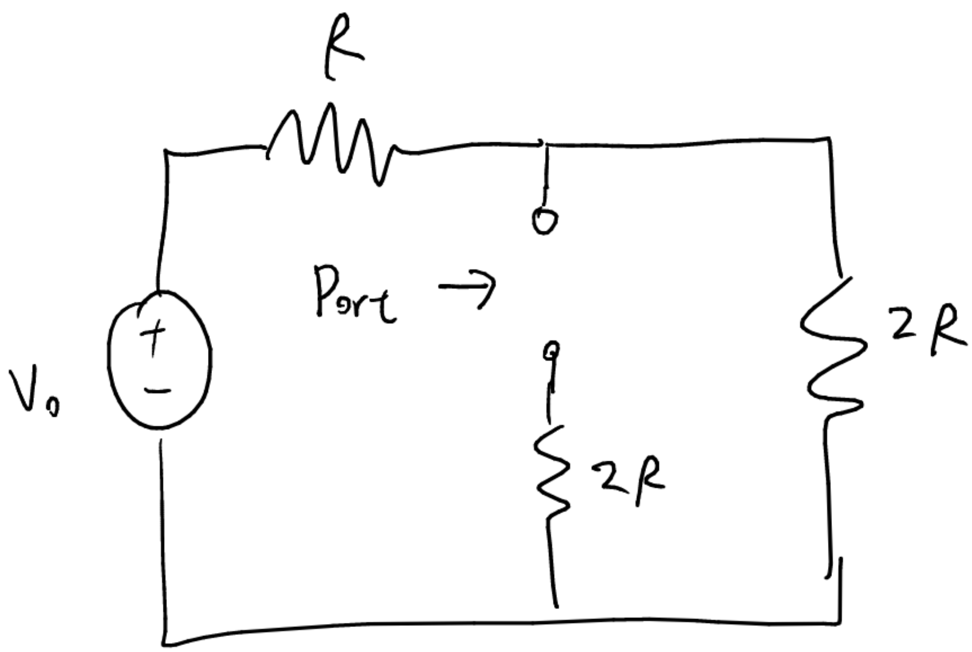
Obviously, the voltage across the inductor $v_L(\infty) = 0$. Current can be solved by whatever circuit analysis technique you prefer, and it comes out to $i_L(\infty) = \frac{V_0}{4R}$.

Final values:

$$ i_L(\infty) = \frac{V_0}{4R} $$ $$ v_L(\infty) = 0 $$

Time constant

The time constant is determined by the inductance L and the resistance viewed from the inductor port. To find this resistance, we can replace the inductor with a port and calculate the Thevenin resistance looking into this port.



Since we're calculating Thevenin resistance, we can turn off the independent source V_0. Then $R_{TH} = 2R + R // 2R = \frac{8R}{3} $.

The time constant is then:

$$ \tau = \frac{L}{R_{TH}} = \frac{L}{(8R)/3} = \frac{3L}{8R} $$.

Putting it all together

$$ i_L(t) = i_L(0^+) e^{-t/\tau} + i_L(\infty) \left(1 - e^{-t/\tau} \right) = \frac{V_0}{R} e^{-t/(\frac{3L}{8R})} + \frac{V_0}{4R} \left(1 - e^{-t/(\frac{3L}{8R})} \right) = \frac{V_0}{4R} + \frac{3V_0}{4R}e^{-t/(\frac{3L}{8R})}$$ $$ v_L(t) = v_L(0^+) e^{-t/\tau} + v_L(\infty) \left(1 - e^{-t/\tau} \right) = (-2V_0) e^{-t/(\frac{3L}{8R})} + (0) \left(1 - e^{-t/(\frac{3L}{8R})} \right) = -2V_0 e^{-t/(\frac{3L}{8R})}$$