Observer

Motivation

In general, we can't know the values of states $\mathbf{q}$ of a state-space system. We only have access to the input x and output y. Observers are used to estimate the values of states q based on the input and output.

A realistic state-space model of a system includes some extra terms:

$$ \mathbf{q}[n+1] = \mathbf{A}\mathbf{q}[n] + \mathbf{b}x[n] + \mathbf{w}[n] $$ $$ y[n] = \mathbf{c}^T\mathbf{q}[n] + dx[n] + \zeta[n] $$

- $\mathbf{w}$ is the system/plant disturbance.
- ζ is the measurement noise.

We can set up a real-time model that is a replica of the real system:

$$ \hat{\mathbf{q}}[n+1] = \mathbf{A}\hat{\mathbf{q}}[n] + \mathbf{b}x[n] $$ $$ \hat{\mathbf{y}}[n] = \mathbf{c}^T\hat{\mathbf{q}}[n] + dx[n] $$

- $\hat{\mathbf{q}} are the states estimated by this model.
- $\hat{\mathbf{y}} is the output estimated by this model.

Notice the differences between the model and the system:

- This model does not have measurement noise ζ.
- This model does not have plant disturbance $\mathbf{w}[n]$.

The error $\tilde{\mathbf{q}}$ is the difference between estimated and actual states.

$$ \tilde{\mathbf{q}} = \mathbf{q} - \hat{\mathbf{q}} $$

The error evolves according to the following equation:

$$ \hat{\mathbf{q}}[n+1] = \mathbf{A}\hat{\mathbf{q}}[n] + w[n] $$

with initial condition:

$$ \tilde{\mathbf{q}}[0] = \mathbf{q}[0] - \hat{\mathbf{q}}[0] $$

Whether the error will go away (i.e. model and physical system converge) depends on the eigenvalues of $\mathbf{A}$.

We can manipulate the system matrix of the error system by adding feedback. Because we can't access the states of the physical plant, we will use the output of the plant.

Observer with feedback

$$ \hat{\mathbf{q}}[n+1] = \mathbf{A}\hat{\mathbf{q}}[n] + \mathbf{b}x[n] - \mathbf{l}(y[n] - \hat{y}[n]) $$

where

- $\mathbf{l}$ is the **observer gain vector**:

$$ \mathbf{l} = \begin{bmatrix} l_1 \\ l_2 \\ \vdots \\ l_L \end{bmatrix} $$

- $y - \hat{y}$ is the output error:

$$ y - \hat{y} = \mathbf{c}^T \tilde{\mathbf{q}} + \zeta $$

Substituting in the output error expression, we get:

$$ \tilde{\mathbf{q}}[n+1] = \mathbf{A} \tilde{\mathbf{q}}[n] + \mathbf{w}[n] + \mathbf{l}\mathbf{c}^T\tilde{\mathbf{q}}[n] + \mathbf{l}\zeta[n] $$ $$ = (\mathbf{A} + \mathbf{l}\mathbf{c}^T) \tilde{\mathbf{q}}[n] + \mathbf{w}[n] + \mathbf{l}\zeta[n] $$

The closed-loop state evolution equation has the system matrix $(\mathbf{A} + \mathbf{l}\mathbf{c}^T)$. We can set $\mathbf{l}$ to make this system stable: i.e. set the eigenvalues such that they have negative real parts (CT case) or magnitudes less than one (DT case). Unfortunately, the tradeoff is that the presence of $\mathbf{l} \neq \mathbf{0}$ gives us the $\mathbf{l}\zeta[n]$ term, which gives us an error based on output measurement noise.

Unobservable modes

As a reminder, an unobservable mode can not be observed in the output. This means that unobservable modes in the plant are also modes of the error system, no matter what $\mathbf{l}$ we choose.

For an unobservable mode associated with eigenvalue λ_k:

$$ \mathbf{A}\mathbf{v}_k = \lambda_k \mathbf{v}_k, \underbrace{\mathbf{c}^T\mathbf{v}_k}_{\xi_k} = 0 $$

$$ (\mathbf{A} + \mathbf{l}\mathbf{c}^T)\mathbf{v}_k = \mathbf{A}\mathbf{v}_k = \lambda\mathbf{v}_k $$

- We can't move the eigenvalue λ_k.

The observable modes of the plant can be moved to arbitrary self-conjugate locations by choice of $\mathbf{l}$. This can be done choosing $\mathbf{l}$ such that:

$$ \mathrm{det}(\lambda \mathbf{I} - \mathbf{A} - \mathbf{l}\mathbf{c}^T) = (\lambda - \epsilon_1)(\lambda - \epsilon_2) \dots (\lambda - \epsilon_L) $$

State feedback

The reason we want state values is to implement state feedback.

Consider a system with the following state evolution equation:

$$ \mathbf{q}[n+1] = \mathbf{A} \mathbf{q}[n] + \mathbf{b} x[n] $$

If we know the values of the state variables q, then we can use those in the input x.

$$ x[n] = \mathbf{g}^T \mathbf{q}[n] + p[n] $$

Then, the closed-loop equation becomes:

$$ \mathbf{q}[n+1] = (\mathbf{A} + \mathbf{b} \mathbf{g}^T) \mathbf{q}[n] + \mathbf{b} x[n] + \mathbf{b} p[n] $$

And we have a new system matrix $(\mathbf{A} + \mathbf{b} \mathbf{g}^T)$ with new eigenvalues.

Observer-based state feedback

Since we cannot directly know the states $\mathbf{q}$ of the system, we can use the estimated states from the observer to approximate state feedback.

$$ x = \mathbf{g}^T \hat{\mathbf{q}} + p $$

where $\tilde{\mathbf{q}}$ is the estimated state vector.

The state vector of this system now has twice as many elements: the original state values and the estimated values from the observer.

One choice of the new state vector is:

$$ \begin{bmatrix} \mathbf{q} \\ \hat{\mathbf{q}} \end{bmatrix} $$

Another choice is:

$$ \begin{bmatrix} \mathbf{q} \\ \tilde{\mathbf{q}} \end{bmatrix} $$

where $\tilde{\mathbf{q}} = \mathbf{q} - \hat{\mathbf{q}}$ is the error between the estimated state vector and the true state vector.

Now we can rewrite the input x as:

$$ x = \mathbf{g}^T (\mathbf{q} - \tilde{\mathbf{q}}) + p $$

Using this new state vector, the right side of our new state evolution equation becomes:

$$ \begin{bmatrix} \mathbf{A} + \mathbf{b}\mathbf{g}^T & -\mathbf{b}\mathbf{g}^T \\ \mathbf{0} & \mathbf{A} + \mathbf{l}\mathbf{c}^T \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \tilde{\mathbf{q}} \end{bmatrix} + \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix} p$$

The eigenvalues of a block triangular matrix are union of the eigenvalues of the top left and bottom right matrices.

$$ \lambda \left(\begin{bmatrix} \mathbf{M} & \mathbf{P} \\ \mathbf{0} & \mathbf{N} \end{bmatrix} \right) = \lambda(\mathbf{M}) \cup \lambda(\mathbf{N}) $$

Therefore, the eigenvalues of the whole observer-based state feedback system are the eigenvalues of $\mathbf{A} + \mathbf{b}\mathbf{g}^T$ and $\mathbf{A} + \mathbf{l}\mathbf{c}^T$.

Effect on reachability/observability

State feedback does not affect reachability because it does not change the ability for the input to excite modes.

It is able to induce unobservability.