Power spectral density

The power spectral density S_{xx} is the Fourier transform of the autocorrelation R_{xx} of a wide-sense stationary process x.

For CT process $x(t)$:

$$ S_{xx}(j\omega) \leftrightarrow R_{xx}(\tau) $$

For DT process $x[n]$:

$$ S_{xx}(e^{j\Omega}) \leftrightarrow R_{xx}[m] $$

Instantaneous power

Instantaneous power is defined:

$$ x^2(t) $$

The expectation of instant power is the autocorrelation with zero time shift:

$$ E[x^2(t)] = R_{xx}(0) $$

The expectation of instantaneous power can be written in terms of the power spectral density:

$$ E[x^2(t)] = R_{xx}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(j\omega) d\omega $$

Therefore, S_{xx} describes how instantaneous power is distributed across frequency.

PSD of filtered process

Consider a process y, which is the WSS process x filtered by a function h:

$$ y(t) = (h \ast x)(t) $$

Then, the PSD of this new process is:

$$ S_{yy}(j\omega) = \left|H(j\omega)\right|^2 S_{xx}(j\omega) $$

Fluctuation spectral density

Fluctuation spectral density is the power spectral density of the fluctuation of a process from its mean.

In other words, it is the Fourier transform of autocovariance.

$$ C_xx[m] \leftrightarrow D_{xx}(e^{j\Omega}) $$ $$ C_xx(\tau) \leftrightarrow D_{xx}(j\omega) $$

White process

A white process has a flat power spectral density. For a white process $x(t)$:

$$ S_{xx}(j\omega) = k, -\infty \lt \omega \lt \infty $$

Energy spectral density

Let $x(t)$ be a random process. Window this signal between $-T$ and T to obtain $x_T(t)$. $x_T(t)$ can also be written as:

$$ x_T(t) = w_T(t)x(t) $$

where $w_T(t) = 1$ for $|t| < T$ and 0 otherwise.

The energy spectral density (ESD) is the square of the Fourier transform of the windowed signal $x_T(t)$:

$$ \left| X_T(j\omega) \right|^2 $$

The ESD has units “energy/Hz.”

Periodogram

The periodogram is defined by the ESD divided by the time interval $2T$.:

$$ \frac{1}{2T} \left| X_T(j\omega) \right|^2 $$

The periodogram has units “power/Hz.”

The limit of the expectation of the periodogram as $T \to \infty$ is the power spectral density:

$$ S_{xx}(j\omega) = \lim_{T \to \infty} \frac{1}{2T} E[|X_T(j\omega)|^2] $$

This result is the Einstein-Wiener-Khinchin theorem.

Spectral estimation

Spectral estimation is estimating power spectral density $S_{xx}(j\omega)$ or cross spectral density $S_{xy}(j\omega)$ from experimental or simulated data.

To do this, we replace the expectation $E[|X_T(j\omega)|^2]$ in the previous section with the average over many iterations from experiments or simulations.