Time-averaging circuit variables

Finding the time average of circuit variables is commonly done in power electronics. It is usually very difficult to calculate time averages explicitly by finding time-varying expressions for the variable in questions and integrating. Instead, we can use the constitutive relations of capacitors and/or inductors to greatly simplify the math.

Recall the constitutive relations of capacitors and inductors:

$$ i_C(t) = C \frac{dv_C}{dt} $$ $$ v_L(t) = L \frac{di_L}{dt} $$

Let's assume that whatever circuit we're working with is in cyclic steady state. (This is usually given.) Then, what happens when we calculate the time average of $i_C(t)$ over a single period from $t = 0$ to $t = T$?

$$ <i_C> = \frac{1}{T} \int_0^T i_C(t) dt $$ $$ = \frac{1}{T} \int_0^T C\frac{dv_C}{dt} dt$$ $$ = \frac{1}{T} C(v_C(T) - v_C(0)) = 0$$ Because we assumed that the circuit is in cyclic steady state, $v_C(T) = v_C(0)$. (More generally, $v_C(t + T) = v_C(t)$.)

We can do similar math to find the time average of $v_L(t)$ over a single period:

$$ <v_L> = \frac{1}{T} \int_0^T v_L(t) dt $$ $$ = \frac{1}{T} \int_0^T L\frac{di_L}{dt} dt$$ $$ = \frac{1}{T} L(i_L(T) - i_L(0)) = 0$$

So now we know, that in cyclic steady state,

$$ <i_C> = 0 $$ $$ <v_L> = 0 $$

How do we use these relations to solve for the time average of other circuit variables? Let's look at an example.

Example (from Fall 2018 Quiz 2 Problem 2)

Consider the following RL circuit. A square wave voltage with period $40ms$ and duty cycle 50% is applied. Numerically determine the average value of the current i_m in the cyclic steady state.

The problem is asking for the time average of $i_m(t)$. We could solve for $<i_m>$ by solving for $<v_R>$ and dividing by R_m (Ohm's law). How can we find $<v_m>$? Let's write out the KVL equation for this circuit:

$$ v_IN(t) - v_L(t) - v_R(t) = 0$$

$$ v_R(t) = v_{IN}(t) - v_L(t) $$

We can take the time average of each term in this equation, and the resulting equation will still be valid (because time-averaging is a linear operation):

$$ <v_R> = <v_{IN}> - <v_L> $$

And as we found above, $<v_L> = 0$.

$$ <v_R> = <v_{IN}> - 0 = <v_{IN}> $$

$$ <v_{IN}> = 5V \times 50\% = 2.5V $$

$$ <i_m> = \frac{2.5V}{R} $$

Much easier than integrating a bunch of exponential decay functions.