

# Zero-input response

A zero-input response (ZIR), or the undriven response, of a [state-space system](#) is its output when the input  $\mathbf{x} = 0$ . In other words, the ZIR is the response of the system to its initial conditions.

## CT case

To find the ZIR of a CT state-space system:

- Set the input to zero:

$$\mathbf{x} \equiv 0$$

- Then, solve the state evolution equation (the second term disappears because the input is zero.):

$$\dot{\mathbf{q}}(t) = \mathbf{A}\mathbf{q}(t)$$

A possible solution is:

$$\mathbf{q}(t) = \mathbf{v}e^{\lambda t}$$

When we plug it into the state evolution equation, we get:

$$e^{\lambda t} \lambda \mathbf{v} = \mathbf{A}\mathbf{v} e^{\lambda t}$$

This tells us that any possible  $\lambda$  is an eigenvalue of the matrix that has the associated eigenvector  $\mathbf{v}$ .

## DT case

To find the ZIR of a DT state-space system:

- Set the input to zero:

$$\mathbf{x} \equiv 0$$

- Then, solve the state evolution equation (the second term disappears because the input is zero.):

$$\mathbf{q}[n+1] = \mathbf{A}\mathbf{q}[n]$$

A possible solution is:  $\mathbf{q}(t) = \lambda^n \mathbf{v}$

When we plug it into the state evolution equation, we get:

$$\lambda^{n+1} \mathbf{v} = \lambda^n \mathbf{A} \mathbf{v}$$

We can simplify this to:

$$\lambda \mathbf{v} = \mathbf{A} \mathbf{v}$$

This is the same as the CT case! This tells us that any possible  $\lambda$  is an eigenvalue of the matrix that has the associated eigenvector  $\mathbf{v}$ .

## Finding eigenvalues and eigenvectors

Any  $\lambda$  and  $\mathbf{v} \neq \mathbf{0}$  that satisfy the above equation is an eigenvalue/eigenvector pair.

Let's now try to find these eigenvalue/eigenvector pairs. We can rewrite the above equation as:

$(\lambda \mathbf{I} - \mathbf{A}) \mathbf{v} = \mathbf{0}$  For any value of  $\lambda$ , if  $(\lambda \mathbf{I} - \mathbf{A})$  is invertible (in other words, if  $\det(\lambda \mathbf{I} - \mathbf{A}) \neq 0$ ), then  $\mathbf{v} = \mathbf{0}$ , so that  $\lambda$  is not an eigenvalue. To find eigenvalues, we can set  $\det(\lambda \mathbf{I} - \mathbf{A}) = 0$ .

- The polynomial  $\det(\lambda \mathbf{I} - \mathbf{A}) = \lambda^L + a_{L-1} \lambda^{L-1} + \dots + a_0$  is known as the characteristic polynomial  $a(\lambda)$ . Setting the  $a(\lambda) = 0$  gives us the characteristic equation, whose roots are the eigenvalues.

Using the characteristic equation, we can find the eigenvalues and eigenvectors.

Solutions to the state evolution equation have the following form:

$$\mathbf{q}(t) = \alpha_1 \mathbf{v}_1 e^{\lambda_1 t} + \alpha_2 \mathbf{v}_2 e^{\lambda_2 t} + \dots + \alpha_L \mathbf{v}_L e^{\lambda_L t}$$

- where  $\mathbf{v}_i$  is the eigenvector associated with eigenvalue  $\lambda_i$ , and  $\alpha_i$  is a constant.

For an eigenvalue  $\lambda$ , the associated eigenvector  $\mathbf{v}$  can be found by solving:

$$(\lambda \mathbf{I} - \mathbf{A}) \mathbf{v} = \mathbf{0}$$

for a solution  $\mathbf{v} \neq \mathbf{0}$ .

## Modal solutions

In summary, the modal solutions of a CT system ZIR is:

$$\mathbf{q}(t) = \sum_{i=1}^L \alpha_i \mathbf{v}_i e^{\lambda_i t}$$

Where the weights  $\alpha_i$  are determined by the initial conditions:

$$\mathbf{q}(0) = \sum_{i=1}^L \alpha_i \mathbf{v}_i$$

The initial conditions are a linear combination of the eigenvectors.

## Asymptotic stability

Asymptotic stability is when the modal solutions to a system for any initial conditions go to zero.

### CT case

If  $\operatorname{Re}\{\lambda_i\} < 0$  for all  $i$ , then  $\mathbf{q}(t) \rightarrow \mathbf{0}$  for any initial conditions  $\mathbf{q}(0)$ .

In other words, all eigenvalues/poles must be on the left-half plane on the complex s-plane.

### DT case

If  $|\lambda_i| < 1$  for all  $i$ , then  $\mathbf{q}[n] \rightarrow \mathbf{0}$  for any initial conditions  $\mathbf{q}[0]$ .

In other words, all eigenvalues/poles must be inside the unit circle on the complex z-plane.

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Last update: **2024-04-30 04:03**